Announcements

1) Scholarship Application,
   $\underbrace{\text{Math Dept}}$ + Al Turfe
   on Dept. Website, due 4/3

2) Job candidate talk on
   Monday (3–4?)

3) Fixed proof from last
   time, up on CTools

Proposition: (closed props)

Let X be a metric space and I an index set.

a) If $\{C_i\}_{i \in I}$ is a collection of closed subsets of X, then

$$\bigcap_{i \in I} C_i \text{ is closed}$$

b) If $C_1, C_2, \ldots, C_n$ are closed subsets of X, then

$$\bigcup_{i=1}^{n} C_i \text{ is closed.}$$

Proof:

a) By De Morgan's laws,

$$\left(\bigcap_{i\in I} C_i\right)^c = \bigcup_{i\in I}(C_i^c).$$

C_i closed $\Rightarrow$ C_i^c is open, hence

$\bigcup_{i\in I}(C_i^c)$ is open $\Rightarrow$ its complement

$\bigcap_{i\in I} C_i$ is closed

b) Same, using De Morgan's laws and the fact that finite intersections of open sets are open. $\square$

Example 1:

$[0,1)$ is not closed in $\mathbb{R}$

since 1 is a limit point,

but $1 \notin [0,1)$.

However, $[0,1)$ is not open,

either, because $0 \in [0,1)$

and $\forall\ \varepsilon > 0$, $B(0, \varepsilon) = (-\varepsilon, \varepsilon)$

is not contained in $[0,1)$.

Remark: (subsets of $\mathbb{R}$ that are both open and closed)

$\mathbb{R}$ is open since $\forall$ $x \in \mathbb{R}$, $\varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) = B(x,\varepsilon) \subseteq \mathbb{R}$.

This implies that $\mathbb{R}^c = \emptyset$ is closed.

However, $\mathbb{R}$ is closed as well since every point in $\mathbb{R}$ is a limit point of $\mathbb{R}$ $(B(x,\varepsilon) \cap \mathbb{R} = B(x,\varepsilon))$. This implies that $\emptyset$ is open

If $S \subseteq \mathbb{R}$ is both open and closed, then either $S = \mathbb{R}$ or $S = \emptyset$.

(All with metric $d(x,y) = |x-y|$)

Definition. (isolated point)

A point $x \in S \subseteq X$, where X is a metric space is called an isolated point for S if $\exists\ \varepsilon > 0$ with

$$B(x, \varepsilon) \cap S = \{x\}.$$

Example 2: $(\mathbb{R})$

Let $S = [0,1] \cup \{3\}$

Then 3 is an isolated point of S since

$B(3,1) = (3-1, 3+1) = (2,4)$

intersects S in only $\{3\}$.

Remark: Since the notion of isolated point is the negation of limit point (for points in S!), any point in S is either a limit point or an isolated point, inexclusively.

Definition. (closure, interior)

The closure of a set S contained in a metric space X is the intersection of all closed sets containing S.

The interior of a set $S \subseteq X$ is the union of all open sets contained in S.

Example 2: $(\mathbb{R})$

$$S = [0, 1)$$

Closure of $S = [0, 1]$

interior of $S = (0, 1)$

Theorem: (closure / limit pts)

The closure of a set $S \subseteq X$ a metric space is equal to the union of S with its limit points.

Notation: limit points of $S = S'$

closure of $S = \overline{S}$

Theorem states: $\boxed{\overline{S} = S \cup S'}$

proof: First, we prove $S \cup S'$ is closed. From there it immediately follows that $S \cup S' \supseteq \overline{S}$ ← smallest closed set containing S.

Set $T = S \cup S'$ and let x be a limit point of T.

Want to show: $x \in T$

Let $\varepsilon > 0$. Since $x \in T'$,

$B(x, \varepsilon) \cap T$ contains points of T other than x.

If $B(x,\varepsilon) \cap S$ contains points of S other than x, we are fine since then $x \in S' \Rightarrow x \in S \cup S' = T$.

Let $y \neq x$, $y \in T \cap B(x,\varepsilon)$.

If $y \in S$, we're done

If $y \notin S$, then $y \in S'$.

Let $\delta = \min\left\{ \frac{d(x,y)}{2}, \frac{\varepsilon - d(x,y)}{2} \right\}$.

Then $B(y,\delta) \subseteq B(x,\varepsilon)$

This implies that
$x \in S'$ since $B(y, \delta) \subseteq B(x, \varepsilon)$
contains points in S that
are not equal to x or y.

This shows $T = S \cup S'$
is closed, hence

$$S \cup S' \supseteq \overline{S}.$$

To show the other direction,
we know that $S \subseteq \overline{S}$, so
we must show that $S' \subseteq \overline{S}$.

Let $x \in S'$. Want: $x \in \overline{S}$.

$\overline{S} = \bigcap_{\substack{C \text{ closed} \\ C \supseteq S}} C$. Let $\varepsilon > 0$.

If $B(x,\varepsilon) \cap \overline{S}$ contains a point other than x is $\overline{S}$, we are done. But $x \in S'$, so in fact $B(x,\varepsilon) \cap \overline{S} \supseteq B(x,\varepsilon) \cap S$, which contains points in $S \subseteq \overline{S}$ not equal to x. Therefore $x \in (\overline{S})' \subseteq \overline{S}$

This shows $S \cup S' = \overline{S}$ $\square$

Definition: (Compactness)

A subset S of a metric space X is **compact** if $(x_n)_{n=1}^{\infty} \subseteq S \Rightarrow (x_n)_{n=1}^{\infty}$ has a convergent subsequence whose limit is in S.

Theorem: (Heine-Borel)

In $\mathbb{R}$ (or $\mathbb{R}^n$) with the usual metric, a subset S is compact if and only if it is closed and bounded.